Remembering that $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\left(\theta\right)$ and $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\left(\theta\right)$, the following matrix is the three-dimensional Givens Rotation from the y -axis to the z -axis (rotation counter-clockwise around the x -axis):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z} + 90^\circ\right) \\ 0 & \sin\left(\theta_{y\to z}\right) & \sin\left(\theta_{y\to z} + 90^\circ\right) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z} + \frac{\pi}{2}\right) \\ 0 & \sin\left(\theta_{y\to z}\right) & \sin\left(\theta_{y\to z} + \frac{\pi}{2}\right) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\to z}\right) & -\sin\left(\theta_{y\to z}\right) \\ 0 & \sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) \end{pmatrix}$$

This matrix rotates three-dimensional column vectors about the origin of the $y \to z$ plane:

$y \to z$	x	y	z
x	1	0	0
y	0	$\cos\left(\theta_{y\to z}\right)$	$-\sin\left(\theta_{y\to z}\right)$
z	0	$\sin\left(\theta_{y\to z}\right)$	$\cos\left(\theta_{y\to z}\right)$

Similarly, the following is the three-dimensional Givens Rotation from the x -axis to the z -axis (rotation clockwise around the y -axis):

$$\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z} + 90^\circ\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z} + 90^\circ\right) \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z} + \frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z} + \frac{\pi}{2}\right) \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & -\sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{pmatrix}$$

This matrix rotates three-dimensional column vectors about the origin of the $x \to z$ plane:

$x \to z$	x	y	z
x	$\cos\left(\theta_{x\to z}\right)$	0	$-\sin\left(\theta_{x\to z}\right)$
y	0	1	0
z	$\sin\left(\theta_{x\to z}\right)$	0	$\cos\left(\theta_{x\to z}\right)$

Notice that this is **not** the same as the traditional rotation matrix where we rotate around the y axis using the right-hand rule (i.e. where we rotate in the opposite direction about the origin of the $x \to z$ plane). In other words, given $\theta = -\phi$ and remembering that $\cos\left(-\phi\right) = \cos\left(\phi\right)$ and $\sin\left(-\phi\right) = -\sin\left(\phi\right)$, we have the following:

$$\begin{pmatrix} \cos\left(\theta\right) & 0 & \cos\left(\theta + 90^\circ\right) \\ 0 & 1 & 0 \\ \sin\left(\theta\right) & 0 & \sin\left(\theta + 90^\circ\right) \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(\left(-\phi\right)\right) & 0 & \cos\left(\left(-\phi\right) + 90^\circ\right) \\ 0 & 1 & 0 \\ \sin\left(\left(-\phi\right)\right) & 0 & \sin\left(\left(-\phi\right) + 90^\circ\right) \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(-\phi\right) & 0 & \sin\left(-\phi + \frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ \sin\left(-\phi\right) & 0 & \sin\left(-\phi + \frac{\pi}{2}\right) \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(-\phi\right) & 0 & -\sin\left(-\phi\right) \\ 0 & 1 & 0 \\ \sin\left(-\phi\right) & 0 & \cos\left(-\phi\right) \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(\phi\right) & 0 & \sin\left(\phi\right) \\ 0 & 1 & 0 \\ -\sin\left(\phi\right) & 0 & \cos\left(\phi\right) \end{pmatrix}$$

Lastly, the following is the three-dimensional Givens Rotation from the x-axis to the y-axis (rotation counter-clockwise around the z-axis):

$$\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y} + 90°\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y} + 90°\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y} + \frac{\pi}{2}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y} + \frac{\pi}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & -\sin\left(\theta_{x\to y}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix rotates three-dimensional column vectors about the origin of the $x \to y$ plane:

$x \to y$	x	y	z
x	$\cos\left(\theta_{x\to y}\right)$	$-\sin\left(\theta_{x\to y}\right)$	0
y	$\sin\left(\theta_{x\to y}\right)$	$\cos\left(\theta_{x\to y}\right)$	0
z	0	0	1

Observe that given n dimensions, we have the following number of distinct Givens Rotation matrices:

$$\left(n-1\right) + \left(n-2\right) + \cdots + 1$$

$$\sum_{k=1}^{n-1} k$$

$$\frac{\left(n-1\right)\left(\left(n-1\right)+1\right)}{2}$$

$$\frac{\left(n-1\right)\left(n-1+1\right)}{2}$$

$$\frac{\left(n-1\right)\left(n\right)}{2}$$

$$\frac{n\left(n-1\right)}{2}$$

For instance, when we're in three dimensions, we have $n = 3$ and $\frac{n\left(n-1\right)}{2} = \frac{\left(3\right)\left(\left(3\right)-1\right)}{2} = \frac{3\left(\cancel{2}\right)}{\cancel{2}} = 3$ distinct Givens Rotation matrices:

$y \to z$	x	y	z
x	1	0	0
y	0	$\cos(\theta_{y \to z})$	$-\sin(\theta_{y \to z})$
z	0	$\sin(\theta_{y \to z})$	$\cos(\theta_{y \to z})$

$x \to z$	x	y	z
x	$\cos(\theta_{x \to z})$	0	$-\sin(\theta_{x \to z})$
y	0	1	0
z	$\sin(\theta_{x \to z})$	0	$\cos(\theta_{x \to z})$

and

$x \to y$	x	y	z
x	$\cos(\theta_{x \to y})$	$-\sin(\theta_{x \to y})$	0
y	$\sin(\theta_{x \to y})$	$\cos(\theta_{x \to y})$	0
z	0	0	1

Moreover, notice the order in which we're considering these distinct Givens Rotation matrices:

	x	y	z
x		$x \to y$	$x \to z$
y			$y \to z$
z			

	x	y	z
x		3	2
y			1
z			

We'll leverage this order in a moment. Before that, let's see what happens when we rotate the unit column vector $(1,0,0)$ using each of these Givens Rotations:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z} + 90^\circ\right) \\ 0 & \sin\left(\theta_{y\to z}\right) & \sin\left(\theta_{y\to z} + 90^\circ\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z} + 90^\circ\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z} + 90^\circ\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z} + \frac{\pi}{2}\right) \\ 0 & \sin\left(\theta_{y\to z}\right) & \sin\left(\theta_{y\to z} + \frac{\pi}{2}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z} + \frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z} + \frac{\pi}{2}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\to z}\right) & -\sin\left(\theta_{y\to z}\right) \\ 0 & \sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & -\sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \cos\left(\theta_{x\to z}\right) \\ 0 \\ \sin\left(\theta_{x\to z}\right) \end{pmatrix}$$

and

$$\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y} + 90^\circ\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y} + 90^\circ\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y} + \frac{\pi}{2}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y} + \frac{\pi}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & -\sin\left(\theta_{x\to y}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(\theta_{x\to y}\right) \\ \sin\left(\theta_{x\to y}\right) \\ 0 \end{pmatrix}$$

As expected, this simply extracts the first column vector from each Givens Rotation matrix. Because we're extracting the first column vector, we'll only see θ in those vectors … we'll never see $\theta + 90^\circ$; so for the Givens Rotation matrices that modify $\left(1,0,0\right)$, $\cos\left(\theta\right)$ will always be in the x position and $\sin\left(\theta\right)$ will walk its way up the remaining dimensions based on the order established above. For the Givens Rotation matrices that do not modify $\left(1,0,0\right)$ (it's just that first matrix in three dimensions), those rotations do not impact the x position.

Looking back at our established order:

	x	y	z
x		3	2

y	1
z	

we see that the non-x rows cannot impact the x position. In this three-dimensional case, we see that the Givens Rotation matrix associated with the y row cannot modify the x position. To make this a little clearer, let's rotate the arbitrarily positioned column vector (x, y, z):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{y\to z}) & \cos(\theta_{y\to z} + 90^\circ) \\ 0 & \sin(\theta_{y\to z}) & \sin(\theta_{y\to z} + 90^\circ) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{y\to z}) & \cos\left(\theta_{y\to z} + \frac{\pi}{2}\right) \\ 0 & \sin(\theta_{y\to z}) & \sin\left(\theta_{y\to z} + \frac{\pi}{2}\right) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{y\to z}) & -\sin(\theta_{y\to z}) \\ 0 & \sin(\theta_{y\to z}) & \cos(\theta_{y\to z}) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y\cos(\theta_{y\to z}) - z\sin(\theta_{y\to z}) \\ y\sin(\theta_{y\to z}) + z\cos(\theta_{y\to z}) \end{pmatrix},$$

$$\begin{pmatrix} \cos(\theta_{x\to z}) & 0 & \cos(\theta_{x\to z} + 90^\circ) \\ 0 & 1 & 0 \\ \sin(\theta_{x\to z}) & 0 & \sin(\theta_{x\to z} + 90^\circ) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to z}) & 0 & \cos\left(\theta_{x\to z} + \frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ \sin(\theta_{x\to z}) & 0 & \sin\left(\theta_{x\to z} + \frac{\pi}{2}\right) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to z}) & 0 & -\sin(\theta_{x\to z}) \\ 0 & 1 & 0 \\ \sin(\theta_{x\to z}) & 0 & \cos(\theta_{x\to z}) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x\cos(\theta_{x\to z}) - z\sin(\theta_{x\to z}) \\ y \\ x\sin(\theta_{x\to z}) + z\cos(\theta_{x\to z}) \end{pmatrix},$$

and

$$\begin{pmatrix} \cos(\theta_{x\to y}) & \cos(\theta_{x\to y} + 90^\circ) & 0 \\ \sin(\theta_{x\to y}) & \sin(\theta_{x\to y} + 90^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to y}) & \cos\left(\theta_{x\to y} + \frac{\pi}{2}\right) & 0 \\ \sin(\theta_{x\to y}) & \sin\left(\theta_{x\to y} + \frac{\pi}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to y}) & -\sin(\theta_{x\to y}) & 0 \\ \sin(\theta_{x\to y}) & \cos(\theta_{x\to y}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x\cos(\theta_{x\to y}) - y\sin(\theta_{x\to y}) \\ x\sin(\theta_{x\to y}) + y\cos(\theta_{x\to y}) \\ z \end{pmatrix}.$$

We immediately see that rotating (x, y, z) about the $y \to z$ plane (multiplying by the first Givens Rotation matrix) does not impact the x position … and only impacts the y and z positions if $y \neq 0$ or $z \neq 0$ … which only happens when we have a column vector off of the x-axis. For the column vector $\left(r,0,0\right)$ on the x-axis, rotating about the $y \rightarrow z$ plane (multiplying by that first Givens Rotation matrix) has no effect … we'll still have $\left(r,0,0\right)$ after that multiplication.

This brings up the question: given the column vector $\left(r,0,0\right)$ where $r = \sqrt{x^2 + y^2 + z^2}$ and $r \geq 0$, can all column vectors $\left(x,y,z\right)$ be recovered using just these Givens Rotation matrices?

First, we'll start by using all the Givens Rotation matrices in our established order … and notice that the first Givens Rotation matrix doesn't really help out that much:

$$\overbrace{\begin{pmatrix} \cos\left(\theta_{x\rightarrow y}\right) & -\sin\left(\theta_{x\rightarrow y}\right) & 0 \\ \sin\left(\theta_{x\rightarrow y}\right) & \cos\left(\theta_{x\rightarrow y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{3} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\rightarrow z}\right) & 0 & -\sin\left(\theta_{x\rightarrow z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\rightarrow z}\right) & 0 & \cos\left(\theta_{x\rightarrow z}\right) \end{pmatrix}}^{2} \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\rightarrow z}\right) & -\sin\left(\theta_{y\rightarrow z}\right) \\ 0 & \sin\left(\theta_{y\rightarrow z}\right) & \cos\left(\theta_{y\rightarrow z}\right) \end{pmatrix}}^{1} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} \cos\left(\theta_{x\rightarrow y}\right) & -\sin\left(\theta_{x\rightarrow y}\right) & 0 \\ \sin\left(\theta_{x\rightarrow y}\right) & \cos\left(\theta_{x\rightarrow y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{3} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\rightarrow z}\right) & 0 & -\sin\left(\theta_{x\rightarrow z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\rightarrow z}\right) & 0 & \cos\left(\theta_{x\rightarrow z}\right) \end{pmatrix}}^{2} \left(\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\rightarrow z}\right) & -\sin\left(\theta_{y\rightarrow z}\right) \\ 0 & \sin\left(\theta_{y\rightarrow z}\right) & \cos\left(\theta_{y\rightarrow z}\right) \end{pmatrix}}^{1} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} \cos\left(\theta_{x\rightarrow y}\right) & -\sin\left(\theta_{x\rightarrow y}\right) & 0 \\ \sin\left(\theta_{x\rightarrow y}\right) & \cos\left(\theta_{x\rightarrow y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{3} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\rightarrow z}\right) & 0 & -\sin\left(\theta_{x\rightarrow z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\rightarrow z}\right) & 0 & \cos\left(\theta_{x\rightarrow z}\right) \end{pmatrix}}^{2} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then, we have:

$$\overbrace{\begin{pmatrix} \cos\left(\theta_{x\rightarrow y}\right) & -\sin\left(\theta_{x\rightarrow y}\right) & 0 \\ \sin\left(\theta_{x\rightarrow y}\right) & \cos\left(\theta_{x\rightarrow y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{3} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\rightarrow z}\right) & 0 & -\sin\left(\theta_{x\rightarrow z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\rightarrow z}\right) & 0 & \cos\left(\theta_{x\rightarrow z}\right) \end{pmatrix}}^{2} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} \cos\left(\theta_{x\rightarrow y}\right)\cos\left(\theta_{x\rightarrow z}\right) & -\sin\left(\theta_{x\rightarrow y}\right) & -\cos\left(\theta_{x\rightarrow y}\right)\sin\left(\theta_{x\rightarrow z}\right) \\ \sin\left(\theta_{x\rightarrow y}\right)\cos\left(\theta_{x\rightarrow z}\right) & \cos\left(\theta_{x\rightarrow y}\right) & -\sin\left(\theta_{x\rightarrow y}\right)\sin\left(\theta_{x\rightarrow z}\right) \\ \sin\left(\theta_{x\rightarrow z}\right) & 0 & \cos\left(\theta_{x\rightarrow z}\right) \end{pmatrix} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} r\cos\left(\theta_{x\rightarrow y}\right)\cos\left(\theta_{x\rightarrow z}\right) \\ r\sin\left(\theta_{x\rightarrow y}\right)\cos\left(\theta_{x\rightarrow z}\right) \\ r\sin\left(\theta_{x\rightarrow z}\right) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For $r = \sqrt{x^2 + y^2 + z^2}$, $r = 0$ implies $x = 0$, $y = 0$, and $z = 0$, so given $r = 0$, we have:

$$\begin{aligned}
\begin{pmatrix} r\cos\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) \\ r\sin\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) \\ r\sin\left(\theta_{x\to z}\right) \end{pmatrix} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
\begin{pmatrix} \left(0\right)\cos\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) \\ \left(0\right)\sin\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) \\ \left(0\right)\sin\left(\theta_{x\to z}\right) \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

So, we can recover the column vector $\left(0,0,0\right)$.

Given $r = \sqrt{x^2 + y^2 + z^2}$ with $r > 0$ and $-90° \leq \theta_{x\to z} \leq 90°$, we have:

$$\begin{aligned}
r\sin\left(\theta_{x\to z}\right) &= z \\
\left(\sqrt{x^2 + y^2 + z^2}\right)\sin\left(\theta_{x\to z}\right) &= z \\
\sin\left(\theta_{x\to z}\right) &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\
\theta_{x\to z} &= \arcsin\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)
\end{aligned}$$

Moreover, since $\arcsin\left(w\right) = \arctan\left(\frac{w}{\sqrt{1 - w^2}}\right)$, we also have:

$$
\begin{aligned}
\theta_{x\to z} &= \arcsin\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\
&= \arctan\left(\frac{\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)}{\sqrt{1-\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)^2}}\right) \\
&= \arctan\left(\frac{\frac{z}{\sqrt{x^2+y^2+z^2}}}{\sqrt{\frac{x^2+y^2+z^2}{x^2+y^2+z^2}-\frac{z^2}{x^2+y^2+z^2}}}\right) \\
&= \arctan\left(\frac{z}{\sqrt{x^2+y^2+z^2}\sqrt{\frac{x^2+y^2+\cancel{z^2}-\cancel{z^2}}{x^2+y^2+z^2}}}\right) \\
&= \arctan\left(\frac{z}{\cancel{\sqrt{x^2+y^2+z^2}}\frac{\sqrt{x^2+y^2}}{\cancel{\sqrt{x^2+y^2+z^2}}}}\right) \\
&= \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)
\end{aligned}
$$

Remember: $\arctan\left(\frac{y}{x}\right)$ is `atan2( y, x )` in C/C++ and `ArcTan[ x, y ]` in Mathematica.

Given $r=\sqrt{x^2+y^2+z^2}$ with $r>0$, $-90^\circ \le \theta_{x\to z} \le 90^\circ$, and remembering that $\cos\left(\arctan\left(w\right)\right)=\frac{1}{\sqrt{1+w^2}}$ and $\sin\left(\arctan\left(w\right)\right)=\frac{w}{\sqrt{1+w^2}}$, if $\theta_{x\to z}=\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)$, we have:

$$
\begin{aligned}
\cos\left(\theta_{x\to z}\right) \\
\boxed{\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right)} \\
\frac{1}{\sqrt{1+\left(\frac{z}{\sqrt{x^2+y^2}}\right)^2}} \\
\frac{1}{\sqrt{\frac{x^2+y^2}{x^2+y^2}+\frac{z^2}{x^2+y^2}}} \\
\frac{1}{\left(\frac{\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}\right)} \\
\boxed{\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}}
\end{aligned}
$$

and

$$
\begin{aligned}
\sin\left(\theta_{x\to z}\right) \\
\boxed{\sin\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right)} \\
\frac{\left(\frac{z}{\sqrt{x^2+y^2}}\right)}{\sqrt{1+\left(\frac{z}{\sqrt{x^2+y^2}}\right)^2}} \\
\frac{\frac{z}{\sqrt{x^2+y^2}}}{\sqrt{\frac{x^2+y^2}{x^2+y^2}+\frac{z^2}{x^2+y^2}}} \\
\frac{z}{\cancel{\sqrt{x^2+y^2}}\,\frac{\sqrt{x^2+y^2+z^2}}{\cancel{\sqrt{x^2+y^2}}}} \\
\boxed{\frac{z}{\sqrt{x^2+y^2+z^2}}}
\end{aligned}
$$

.

Given $r=\sqrt{x^2+y^2+z^2}$ with $r>0$ and $-180^\circ \le \theta_{x\to y} \le 180^\circ$, since

$\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right)=\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, if $\theta_{x\to z}=\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)$, we have:

$$
\begin{aligned}
r\sin\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) &= y \\
r\sin\left(\theta_{x\to y}\right)\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) &= y \\
\left(\sqrt{x^2+y^2+z^2}\right)\sin\left(\theta_{x\to y}\right)\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right) &= y \\
\cancel{\sqrt{x^2+y^2+z^2}}\,\sin\left(\theta_{x\to y}\right)\frac{\sqrt{x^2+y^2}}{\cancel{\sqrt{x^2+y^2+z^2}}} &= y \\
\sin\left(\theta_{x\to y}\right) &= \frac{y}{\sqrt{x^2+y^2}} \\
\theta_{x\to y} &= \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)
\end{aligned}
$$

Moreover, since $\arcsin\left(w\right)=\arctan\left(\frac{w}{\sqrt{1-w^2}}\right)$, we also have:

$$
\begin{aligned}
\theta_{x\to y} &= \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right) \\
&= \arctan\left(\frac{\left(\frac{y}{\sqrt{x^2+y^2}}\right)}{\sqrt{1-\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}}\right) \\
&= \arctan\left(\frac{y}{\sqrt{x^2+y^2}\sqrt{\frac{x^2+y^2}{x^2+y^2}-\frac{y^2}{x^2+y^2}}}\right) \\
&= \arctan\left(\frac{y}{\sqrt{x^2+y^2}\sqrt{\frac{x^2+\cancel{y^2}-\cancel{y^2}}{x^2+y^2}}}\right) \\
&= \arctan\left(\frac{y}{\cancel{\sqrt{x^2+y^2}}\frac{\sqrt{x^2}}{\cancel{\sqrt{x^2+y^2}}}}\right) \\
&= \arctan\left(\frac{y}{\sqrt{x^2}}\right) \\
&= \arctan\left(\frac{y}{x}\right)
\end{aligned}
$$

Given $r=\sqrt{x^2+y^2+z^2}$ with $r>0$, $-180^\circ \le \theta_{x\to y} \le 180^\circ$, and remembering that

$\cos\left(\arctan\left(w\right)\right)=\frac{1}{\sqrt{1+w^2}}$ and $\sin\left(\arctan\left(w\right)\right)=\frac{w}{\sqrt{1+w^2}}$, we have:

$$\cos\left(\theta_{x\to y}\right)$$
$$\boxed{\cos\left(\arctan\left(\frac{y}{x}\right)\right)}$$
$$\frac{1}{\sqrt{1+\left(\frac{y}{x}\right)^2}}$$
$$\frac{1}{\sqrt{\frac{x^2}{x^2}+\frac{y^2}{x^2}}}$$
$$\frac{1}{\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2}}\right)}$$
$$\frac{\sqrt{x^2}}{\sqrt{x^2+y^2}}$$
$$\boxed{\frac{x}{\sqrt{x^2+y^2}}}$$

and

$$\sin\left(\theta_{x\to y}\right)$$
$$\boxed{\sin\left(\arctan\left(\frac{y}{x}\right)\right)}$$
$$\frac{\left(\frac{y}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^2}}$$
$$\frac{\frac{y}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{y^2}{x^2}}}$$
$$\frac{y}{x\frac{\sqrt{x^2+y^2}}{\sqrt{x^2}}}$$
$$\frac{y}{\cancel{x}\frac{\sqrt{x^2+y^2}}{\cancel{x}}}$$
$$\boxed{\frac{y}{\sqrt{x^2+y^2}}}$$

Given $r=\sqrt{x^2+y^2+z^2}$ with $r>0$, $-180^\circ\leq\theta_{x\to y}\leq 180^\circ$, and $-90^\circ\leq\theta_{x\to z}\leq 90^\circ$, since $\cos\left(\arctan\left(\frac{y}{x}\right)\right)=\frac{x}{\sqrt{x^2+y^2}}$ and $\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right)=\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, if $\theta_{x\to y}=\arctan\left(\frac{y}{x}\right)$ and $\theta_{x\to z}=\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)$, we have:

$$r\cos\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right)$$
$$r\cos\left(\arctan\left(\frac{y}{x}\right)\right)\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right)$$
$$\left(\sqrt{x^2+y^2+z^2}\right)\left(\frac{x}{\sqrt{x^2+y^2}}\right)\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right)$$
$$\cancel{\sqrt{x^2+y^2+z^2}}\,\frac{x}{\cancel{\sqrt{x^2+y^2}}}\,\frac{\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2+z^2}}}$$
$$x$$

So, we have $r\cos\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right)=x$.

Obviously, since $\sin\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{y}{\sqrt{x^2+y^2}}$, $\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, and $\sin\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) = \frac{z}{\sqrt{x^2+y^2+z^2}}$, if $\theta_{x\to y} = \arctan\left(\frac{y}{x}\right)$ and $\theta_{x\to z} = \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)$, we also have:

$$\begin{gathered} r\sin\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) \\ r\sin\left(\arctan\left(\frac{y}{x}\right)\right)\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) \\ \left(\sqrt{x^2+y^2+z^2}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)\left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right) \\ \cancel{\sqrt{x^2+y^2+z^2}}\,\frac{y}{\cancel{\sqrt{x^2+y^2}}}\,\frac{\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2+z^2}}} \\ y \end{gathered} \quad \text{and} \quad \begin{gathered} r\sin\left(\theta_{x\to z}\right) \\ r\sin\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) \\ \left(\sqrt{x^2+y^2+z^2}\right)\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right). \\ \cancel{\sqrt{x^2+y^2+z^2}}\,\frac{z}{\cancel{\sqrt{x^2+y^2+z^2}}} \\ z \end{gathered}$$

So, we also have $r\sin\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) = y$ and $r\sin\left(\theta_{x\to z}\right) = z$ such that:

$$\begin{aligned} \begin{pmatrix} r\cos\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) \\ r\sin\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) \\ r\sin\left(\theta_{x\to z}\right) \end{pmatrix} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned}$$

Therefore, all column vectors $\left(x, y, z\right)$ can be recovered by applying the Givens Rotation matrices associated with the x row to the column vector $\left(r, 0, 0\right)$ where $r = \sqrt{x^2+y^2+z^2}$ and $r \geq 0$:

	x	y	z
x		3	2
y			1
z			

Notice that the $\left(r, \theta_{x\to y}, \theta_{x\to z}\right)$ coordinate system implied here is **not** the same as the traditional spherical coordinate system $\left(\rho, \theta, \phi\right)$. In the traditional spherical coordinate system, we have:

$$\begin{aligned} x &= \rho\cos(\theta)\sin(\phi) \\ y &= \rho\sin(\theta)\sin(\phi) \\ z &= \rho\cos(\phi) \end{aligned}$$

where $\rho = \sqrt{x^2 + y^2 + z^2}$, $0^\circ \leq \theta \leq 360^\circ$ where θ is the angle measured from the positive x-axis (toward the positive y-axis), and $0^\circ \leq \phi \leq 180^\circ$ where ϕ is the angle measured from the positive z-axis (toward the negative z-axis); however, what we have here is the following:

$$\begin{aligned} x &= r\cos(\theta_{x\to y})\cos(\theta_{x\to z}) \\ y &= r\sin(\theta_{x\to y})\cos(\theta_{x\to z}) \\ z &= r\sin(\theta_{x\to z}) \end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, $-180^\circ \leq \theta_{x\to y} \leq 180^\circ$ where $\theta_{x\to y}$ is the angle measured from the positive x-axis (toward the positive y-axis), and $-90^\circ \leq \theta_{x\to z} \leq 90^\circ$ where $\theta_{x\to z}$ is the angle measured from the positive x-axis (toward the positive z-axis).

Now, let's rotate a column vector (x, y, z) back to the x-axis so that we get the column vector $(r, 0, 0)$. In order to do that, let's first take a look at each inverse of these Givens Rotation matrices using augmentation and finding the reduced row echelon forms … which as you'll see is very similar for each matrix. Remembering that $\cos^2(\theta) + \sin^2(\theta) = 1$ which implies that $1 - \sin^2(\theta) = \cos^2(\theta)$, we have:

$$\boxed{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\to z}\right) & -\sin\left(\theta_{y\to z}\right) & 0 & 1 & 0 \\ 0 & \sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) & 0 & 0 & 1 \end{array}\right)}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\dfrac{\sin\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{y\to z}\right)} & 0 \\ 0 & \sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\dfrac{\sin\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{y\to z}\right)} & 0 \\ 0 & 0 & \cos\left(\theta_{y\to z}\right) + \dfrac{\sin^2\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 0 & -\dfrac{\sin\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\dfrac{\sin\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{y\to z}\right)} & 0 \\ 0 & 0 & \dfrac{\cos^2\left(\theta_{y\to z}\right) + \sin^2\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 0 & -\dfrac{\sin\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\dfrac{\sin\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{y\to z}\right)} & 0 \\ 0 & 0 & \dfrac{1}{\cos\left(\theta_{y\to z}\right)} & 0 & -\dfrac{\sin\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\dfrac{\sin\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{y\to z}\right)} & 0 \\ 0 & 0 & 1 & 0 & -\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dfrac{1}{\cos\left(\theta_{y\to z}\right)} - \dfrac{\sin^2\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & \sin\left(\theta_{y\to z}\right) \\ 0 & 0 & 1 & 0 & -\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dfrac{1 - \sin^2\left(\theta_{y\to z}\right)}{\cos\left(\theta_{y\to z}\right)} & \sin\left(\theta_{y\to z}\right) \\ 0 & 0 & 1 & 0 & -\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dfrac{\cos^{\cancel{2}}\left(\theta_{y\to z}\right)}{\cancel{\cos\left(\theta_{y\to z}\right)}} & \sin\left(\theta_{y\to z}\right) \\ 0 & 0 & 1 & 0 & -\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) \end{array}\right)$$

$$\boxed{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cos\left(\theta_{y\to z}\right) & \sin\left(\theta_{y\to z}\right) \\ 0 & 0 & 1 & 0 & -\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) \end{array}\right)}$$

$$\boxed{\left(\begin{array}{ccc|ccc} \cos\left(\theta_{x\to z}\right) & 0 & -\sin\left(\theta_{x\to z}\right) & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) & 0 & 0 & 1 \end{array}\right)}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\dfrac{\sin\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & \dfrac{1}{\cos\left(\theta_{x\to z}\right)} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\dfrac{\sin\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & \dfrac{1}{\cos\left(\theta_{x\to z}\right)} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \cos\left(\theta_{x\to z}\right) + \dfrac{\sin^2\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & -\dfrac{\sin\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\dfrac{\sin\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & \dfrac{1}{\cos\left(\theta_{x\to z}\right)} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \dfrac{\cos^2\left(\theta_{x\to z}\right) + \sin^2\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & -\dfrac{\sin\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\dfrac{\sin\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & \dfrac{1}{\cos\left(\theta_{x\to z}\right)} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{\cos\left(\theta_{x\to z}\right)} & -\dfrac{\sin\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\dfrac{\sin\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & \dfrac{1}{\cos\left(\theta_{x\to z}\right)} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \dfrac{1}{\cos\left(\theta_{x\to z}\right)} - \dfrac{\sin^2\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & 0 & \sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \dfrac{1 - \sin^2\left(\theta_{x\to z}\right)}{\cos\left(\theta_{x\to z}\right)} & 0 & \sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \dfrac{\cos^{\cancel{2}}\left(\theta_{x\to z}\right)}{\cancel{\cos\left(\theta_{x\to z}\right)}} & 0 & \sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{array}\right)$$

$$\boxed{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{array}\right)}$$

$$\boxed{\left(\begin{array}{ccc|ccc} \cos\left(\theta_{x\to y}\right) & -\sin\left(\theta_{x\to y}\right) & 0 & 1 & 0 & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\dfrac{\sin\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{x\to y}\right)} & 0 & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -\dfrac{\sin\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{x\to y}\right)} & 0 & 0 \\ 0 & \cos\left(\theta_{x\to y}\right) + \dfrac{\sin^2\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 0 & -\dfrac{\sin\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -\dfrac{\sin\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{x\to y}\right)} & 0 & 0 \\ 0 & \dfrac{\cos^2\left(\theta_{x\to y}\right) + \sin^2\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 0 & -\dfrac{\sin\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -\dfrac{\sin\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{x\to y}\right)} & 0 & 0 \\ 0 & \dfrac{1}{\cos\left(\theta_{x\to y}\right)} & 0 & -\dfrac{\sin\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -\dfrac{\sin\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & 0 & \dfrac{1}{\cos\left(\theta_{x\to y}\right)} & 0 & 0 \\ 0 & 1 & 0 & -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \dfrac{1}{\cos\left(\theta_{x\to y}\right)} - \dfrac{\sin^2\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & \sin\left(\theta_{x\to y}\right) & 0 \\ 0 & 1 & 0 & -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \dfrac{1 - \sin^2\left(\theta_{x\to y}\right)}{\cos\left(\theta_{x\to y}\right)} & \sin\left(\theta_{x\to y}\right) & 0 \\ 0 & 1 & 0 & -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \dfrac{\cos^{\cancel{2}}\left(\theta_{x\to y}\right)}{\cancel{\cos\left(\theta_{x\to y}\right)}} & \sin\left(\theta_{x\to y}\right) & 0 \\ 0 & 1 & 0 & -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\boxed{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y}\right) & 0 \\ 0 & 1 & 0 & -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)}$$

Therefore, the inverse of each three-dimensional Givens Rotation matrix is simply the transpose of that matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{y\to z}) & -\sin(\theta_{y\to z}) \\ 0 & \sin(\theta_{y\to z}) & \cos(\theta_{y\to z}) \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{y\to z}) & -\sin(\theta_{y\to z}) \\ 0 & \sin(\theta_{y\to z}) & \cos(\theta_{y\to z}) \end{pmatrix}^{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{y\to z}) & \sin(\theta_{y\to z}) \\ 0 & -\sin(\theta_{y\to z}) & \cos(\theta_{y\to z}) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to z}) & 0 & -\sin(\theta_{x\to z}) \\ 0 & 1 & 0 \\ \sin(\theta_{x\to z}) & 0 & \cos(\theta_{x\to z}) \end{pmatrix}^{-1} = \begin{pmatrix} \cos(\theta_{x\to z}) & 0 & -\sin(\theta_{x\to z}) \\ 0 & 1 & 0 \\ \sin(\theta_{x\to z}) & 0 & \cos(\theta_{x\to z}) \end{pmatrix}^{T} = \begin{pmatrix} \cos(\theta_{x\to z}) & 0 & \sin(\theta_{x\to z}) \\ 0 & 1 & 0 \\ -\sin(\theta_{x\to z}) & 0 & \cos(\theta_{x\to z}) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to y}) & -\sin(\theta_{x\to y}) & 0 \\ \sin(\theta_{x\to y}) & \cos(\theta_{x\to y}) & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos(\theta_{x\to y}) & -\sin(\theta_{x\to y}) & 0 \\ \sin(\theta_{x\to y}) & \cos(\theta_{x\to y}) & 0 \\ 0 & 0 & 1 \end{pmatrix}^{T} = \begin{pmatrix} \cos(\theta_{x\to y}) & \sin(\theta_{x\to y}) & 0 \\ -\sin(\theta_{x\to y}) & \cos(\theta_{x\to y}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and it shouldn't be too hard to see that the inverse of each n -dimensional Givens Rotation matrix is also the transpose of that matrix … like it is for any other n -dimensional rotation matrix.

Also, if we consider rotations in the opposite direction where $\theta = -\phi$ while remembering that $\cos(-\phi) = \cos(\phi)$ and $\sin(-\phi) = -\sin(\phi)$, then we have:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{y\to z}) & \sin(\theta_{y\to z}) \\ 0 & -\sin(\theta_{y\to z}) & \cos(\theta_{y\to z}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(-\phi_{y\to -z}) & \sin(-\phi_{y\to -z}) \\ 0 & -\sin(-\phi_{y\to -z}) & \cos(-\phi_{y\to -z}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi_{y\to -z}) & -\sin(\phi_{y\to -z}) \\ 0 & \sin(\phi_{y\to -z}) & \cos(\phi_{y\to -z}) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to z}) & 0 & \sin(\theta_{x\to z}) \\ 0 & 1 & 0 \\ -\sin(\theta_{x\to z}) & 0 & \cos(\theta_{x\to z}) \end{pmatrix} = \begin{pmatrix} \cos(-\phi_{x\to -z}) & 0 & \sin(-\phi_{x\to -z}) \\ 0 & 1 & 0 \\ -\sin(-\phi_{x\to -z}) & 0 & \cos(-\phi_{x\to -z}) \end{pmatrix} = \begin{pmatrix} \cos(\phi_{x\to -z}) & 0 & -\sin(\phi_{x\to -z}) \\ 0 & 1 & 0 \\ \sin(\phi_{x\to -z}) & 0 & \cos(\phi_{x\to -z}) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to y}) & \sin(\theta_{x\to y}) & 0 \\ -\sin(\theta_{x\to y}) & \cos(\theta_{x\to y}) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(-\phi_{x\to -y}) & \sin(-\phi_{x\to -y}) & 0 \\ -\sin(-\phi_{x\to -y}) & \cos(-\phi_{x\to -y}) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\phi_{x\to -y}) & -\sin(\phi_{x\to -y}) & 0 \\ \sin(\phi_{x\to -y}) & \cos(\phi_{x\to -y}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We see that each inverse amounts to just rotating in the opposite direction. In other words, since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$ implies that $\cos(\theta) = \cos(-\theta)$ and $\sin(\theta) = -\sin(-\theta)$, we have:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{y\to z}) & -\sin(\theta_{y\to z}) \\ 0 & \sin(\theta_{y\to z}) & \cos(\theta_{y\to z}) \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{y\to z}) & \sin(\theta_{y\to z}) \\ 0 & -\sin(\theta_{y\to z}) & \cos(\theta_{y\to z}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta_{y\to z}) & -\sin(-\theta_{y\to z}) \\ 0 & \sin(-\theta_{y\to z}) & \cos(-\theta_{y\to z}) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to z}) & 0 & -\sin(\theta_{x\to z}) \\ 0 & 1 & 0 \\ \sin(\theta_{x\to z}) & 0 & \cos(\theta_{x\to z}) \end{pmatrix}^{-1} = \begin{pmatrix} \cos(\theta_{x\to z}) & 0 & \sin(\theta_{x\to z}) \\ 0 & 1 & 0 \\ -\sin(\theta_{x\to z}) & 0 & \cos(\theta_{x\to z}) \end{pmatrix} = \begin{pmatrix} \cos(-\theta_{x\to z}) & 0 & -\sin(-\theta_{x\to z}) \\ 0 & 1 & 0 \\ \sin(-\theta_{x\to z}) & 0 & \cos(-\theta_{x\to z}) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to y}) & -\sin(\theta_{x\to y}) & 0 \\ \sin(\theta_{x\to y}) & \cos(\theta_{x\to y}) & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos(\theta_{x\to y}) & \sin(\theta_{x\to y}) & 0 \\ -\sin(\theta_{x\to y}) & \cos(\theta_{x\to y}) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(-\theta_{x\to y}) & -\sin(-\theta_{x\to y}) & 0 \\ \sin(-\theta_{x\to y}) & \cos(-\theta_{x\to y}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, let's use those inverse Givens Rotation matrices to rotate a column vector $\left(x, y, z\right)$ in those opposite directions back to the x-axis so that we get the column vector $\left(r, 0, 0\right)$.

First, we'll start by using all the Givens Rotation matrices in our established order … and get rid of the ones that don't matter here:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & -\sin\left(\theta_{x\to y}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{3} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & -\sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{pmatrix}}^{2} \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\to z}\right) & -\sin\left(\theta_{y\to z}\right) \\ 0 & \sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) \end{pmatrix}}^{1} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & -\sin\left(\theta_{x\to y}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{3} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & -\sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{pmatrix}}^{2} \left(\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\theta_{y\to z}\right) & -\sin\left(\theta_{y\to z}\right) \\ 0 & \sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) \end{pmatrix}}^{1} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & -\sin\left(\theta_{x\to y}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{3} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & -\sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{pmatrix}}^{2} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

Next, remembering that the inverse of a rotation matrix is its transpose, then we have:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & -\sin\left(\theta_{x\to y}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{3} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & -\sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{pmatrix}}^{2} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & -\sin\left(\theta_{x\to y}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}}^{\text{Inverse of 3}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & -\sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{pmatrix}}^{2} \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & -\sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 \\ \sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{pmatrix}^{-1}}^{\text{Inverse of 2}} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & -\sin\left(\theta_{x\to y}\right) & 0 \\ \sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}}^{\text{Inverse of 3}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} \cos\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z}\right) \\ 0 & 1 & 0 \\ -\sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right) \end{pmatrix}}^{\text{Inverse of 2}} \overbrace{\begin{pmatrix} \cos\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y}\right) & 0 \\ -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{\text{Inverse of 3}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

Keeping in mind that $\cos\left(\theta\right) = \cos\left(-\theta\right)$ and $\sin\left(\theta\right) = -\sin\left(-\theta\right)$, we have:

$$\overbrace{\begin{pmatrix}\cos\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z}\right)\\ 0 & 1 & 0\\ -\sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right)\end{pmatrix}}^{\text{Inverse of 2}}\overbrace{\begin{pmatrix}\cos\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y}\right) & 0\\ -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{\text{Inverse of 3}}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

$$\overbrace{\begin{pmatrix}\cos\left(-\theta_{x\to z}\right) & 0 & -\sin\left(-\theta_{x\to z}\right)\\ 0 & 1 & 0\\ \sin\left(-\theta_{x\to z}\right) & 0 & \cos\left(-\theta_{x\to z}\right)\end{pmatrix}}^{\text{Inverse of 2}}\overbrace{\begin{pmatrix}\cos\left(-\theta_{x\to y}\right) & -\sin\left(-\theta_{x\to y}\right) & 0\\ \sin\left(-\theta_{x\to y}\right) & \cos\left(-\theta_{x\to y}\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{\text{Inverse of 3}}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

$$\overbrace{\begin{pmatrix}\cos\left(\left(-\theta_{x\to z}\right)\right) & 0 & -\sin\left(\left(-\theta_{x\to z}\right)\right)\\ 0 & 1 & 0\\ \sin\left(\left(-\theta_{x\to z}\right)\right) & 0 & \cos\left(\left(-\theta_{x\to z}\right)\right)\end{pmatrix}}^{2 \text{ for } -\theta_{x\to z}}\overbrace{\begin{pmatrix}\cos\left(\left(-\theta_{x\to y}\right)\right) & -\sin\left(\left(-\theta_{x\to y}\right)\right) & 0\\ \sin\left(\left(-\theta_{x\to y}\right)\right) & \cos\left(\left(-\theta_{x\to y}\right)\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{3 \text{ for } -\theta_{x\to y}}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

Since $\theta = -\phi$ implies $-\theta = \phi$, we also have:

$$\overbrace{\begin{pmatrix}\cos\left(\left(-\theta_{x\to z}\right)\right) & 0 & -\sin\left(\left(-\theta_{x\to z}\right)\right)\\ 0 & 1 & 0\\ \sin\left(\left(-\theta_{x\to z}\right)\right) & 0 & \cos\left(\left(-\theta_{x\to z}\right)\right)\end{pmatrix}}^{2 \text{ for } -\theta_{x\to z}}\overbrace{\begin{pmatrix}\cos\left(\left(-\theta_{x\to y}\right)\right) & -\sin\left(\left(-\theta_{x\to y}\right)\right) & 0\\ \sin\left(\left(-\theta_{x\to y}\right)\right) & \cos\left(\left(-\theta_{x\to y}\right)\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{3 \text{ for } -\theta_{x\to y}}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

$$\overbrace{\begin{pmatrix}\cos\left(\phi_{x\to -z}\right) & 0 & -\sin\left(\phi_{x\to -z}\right)\\ 0 & 1 & 0\\ \sin\left(\phi_{x\to -z}\right) & 0 & \cos\left(\phi_{x\to -z}\right)\end{pmatrix}}^{2 \text{ for } \phi_{x\to -z}}\overbrace{\begin{pmatrix}\cos\left(\phi_{x\to -y}\right) & -\sin\left(\phi_{x\to -y}\right) & 0\\ \sin\left(\phi_{x\to -y}\right) & \cos\left(\phi_{x\to -y}\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{3 \text{ for } \phi_{x\to -y}}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

In other words, given a column vector $\left(x, y, z\right)$, we can rotate that vector back to the x-axis by applying [Givens Rotation] matrices in the opposite order using the opposite angles. Instead of the following:

	x	y	z
x		$x \to y$	$x \to z$
y			$y \to z$
z			

	x	y	z
x		3	2
y			1
z			

repeated again with subscripts to keep things clear:

	x	y	z
x		3_{forward}	2_{forward}
y			1_{forward}
z			

we now have the following:

	$-x$	$-y$	$-z$
x		$x \to -y$	$x \to -z$
y			$y \to -z$
z			

	$-x$	$-y$	$-z$
x		1_{backward}	2_{backward}
y			3_{backward}
z			

In terms of this new backward ordering and keeping in mind that $\phi = -\theta$, we now have the following:

$$\overbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\left(\phi_{y\to -z}\right) & -\sin\left(\phi_{y\to -z}\right)\\ 0 & \sin\left(\phi_{y\to -z}\right) & \cos\left(\phi_{y\to -z}\right)\end{pmatrix}}^{3_{\text{backward}}\ \text{for}\ \phi_{y\to -z}}\overbrace{\begin{pmatrix}\cos\left(\phi_{x\to -z}\right) & 0 & -\sin\left(\phi_{x\to -z}\right)\\ 0 & 1 & 0\\ \sin\left(\phi_{x\to -z}\right) & 0 & \cos\left(\phi_{x\to -z}\right)\end{pmatrix}}^{2_{\text{backward}}\ \text{for}\ \phi_{x\to -z}}\overbrace{\begin{pmatrix}\cos\left(\phi_{x\to -y}\right) & -\sin\left(\phi_{x\to -y}\right) & 0\\ \sin\left(\phi_{x\to -y}\right) & \cos\left(\phi_{x\to -y}\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{1_{\text{backward}}\ \text{for}\ \phi_{x\to -y}}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

$$\overbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\left(-\theta_{y\to z}\right) & -\sin\left(-\theta_{y\to z}\right)\\ 0 & \sin\left(-\theta_{y\to z}\right) & \cos\left(-\theta_{y\to z}\right)\end{pmatrix}}^{3_{\text{backward}}\ \text{for}\ -\theta_{y\to z}}\overbrace{\begin{pmatrix}\cos\left(-\theta_{x\to z}\right) & 0 & -\sin\left(-\theta_{x\to z}\right)\\ 0 & 1 & 0\\ \sin\left(-\theta_{x\to z}\right) & 0 & \cos\left(-\theta_{x\to z}\right)\end{pmatrix}}^{2_{\text{backward}}\ \text{for}\ -\theta_{x\to z}}\overbrace{\begin{pmatrix}\cos\left(-\theta_{x\to y}\right) & -\sin\left(-\theta_{x\to y}\right) & 0\\ \sin\left(-\theta_{x\to y}\right) & \cos\left(-\theta_{x\to y}\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{1_{\text{backward}}\ \text{for}\ -\theta_{x\to y}}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

$$\overbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\left(-\theta_{y\to z}\right) & -\sin\left(-\theta_{y\to z}\right)\\ 0 & \sin\left(-\theta_{y\to z}\right) & \cos\left(-\theta_{y\to z}\right)\end{pmatrix}}^{1_{\text{forward}}\ \text{for}\ -\theta_{y\to z}}\overbrace{\begin{pmatrix}\cos\left(-\theta_{x\to z}\right) & 0 & -\sin\left(-\theta_{x\to z}\right)\\ 0 & 1 & 0\\ \sin\left(-\theta_{x\to z}\right) & 0 & \cos\left(-\theta_{x\to z}\right)\end{pmatrix}}^{2_{\text{forward}}\ \text{for}\ -\theta_{x\to z}}\overbrace{\begin{pmatrix}\cos\left(-\theta_{x\to y}\right) & -\sin\left(-\theta_{x\to y}\right) & 0\\ \sin\left(-\theta_{x\to y}\right) & \cos\left(-\theta_{x\to y}\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{3_{\text{forward}}\ \text{for}\ -\theta_{x\to y}}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

$$\overbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\left(\theta_{y\to z}\right) & \sin\left(\theta_{y\to z}\right)\\ 0 & -\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right)\end{pmatrix}}^{\text{Inverse of } 1_{\text{forward}}\ \text{for}\ \theta_{y\to z}}\overbrace{\begin{pmatrix}\cos\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z}\right)\\ 0 & 1 & 0\\ -\sin\left(\theta_{x\to z}\right) & 0 & \cos\left(\theta_{x\to z}\right)\end{pmatrix}}^{\text{Inverse of } 2_{\text{forward}}\ \text{for}\ \theta_{x\to z}}\overbrace{\begin{pmatrix}\cos\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y}\right) & 0\\ -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{\text{Inverse of } 3_{\text{forward}}\ \text{for}\ \theta_{x\to y}}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\cos\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z}\right)\\ -\sin\left(\theta_{x\to z}\right)\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) & \cos\left(\theta_{x\to z}\right)\sin\left(\theta_{y\to z}\right)\\ -\sin\left(\theta_{x\to z}\right)\cos\left(\theta_{y\to z}\right) & -\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{x\to z}\right)\cos\left(\theta_{y\to z}\right)\end{pmatrix}\overbrace{\begin{pmatrix}\cos\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y}\right) & 0\\ -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{\text{Inverse of } 3_{\text{forward}}\ \text{for}\ \theta_{x\to y}}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\cos\left(\theta_{x\to z}\right) & 0 & \sin\left(\theta_{x\to z}\right)\\ -\sin\left(\theta_{x\to z}\right)\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{y\to z}\right) & \cos\left(\theta_{x\to z}\right)\sin\left(\theta_{y\to z}\right)\\ -\sin\left(\theta_{x\to z}\right)\cos\left(\theta_{y\to z}\right) & -\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{x\to z}\right)\cos\left(\theta_{y\to z}\right)\end{pmatrix}\overbrace{\begin{pmatrix}\cos\left(\theta_{x\to y}\right) & \sin\left(\theta_{x\to y}\right) & 0\\ -\sin\left(\theta_{x\to y}\right) & \cos\left(\theta_{x\to y}\right) & 0\\ 0 & 0 & 1\end{pmatrix}}^{\text{Inverse of } 3_{\text{forward}}\ \text{for}\ \theta_{x\to y}}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

So, we have:

$$\begin{pmatrix}\cos\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) & \sin\left(\theta_{x\to y}\right)\cos\left(\theta_{x\to z}\right) & \sin\left(\theta_{x\to z}\right)\\ -\cos\left(\theta_{x\to y}\right)\sin\left(\theta_{x\to z}\right)\sin\left(\theta_{y\to z}\right)-\sin\left(\theta_{x\to y}\right)\cos\left(\theta_{y\to z}\right) & -\sin\left(\theta_{x\to y}\right)\sin\left(\theta_{x\to z}\right)\sin\left(\theta_{y\to z}\right)+\cos\left(\theta_{x\to y}\right)\cos\left(\theta_{y\to z}\right) & \cos\left(\theta_{x\to z}\right)\sin\left(\theta_{y\to z}\right)\\ -\cos\left(\theta_{x\to y}\right)\sin\left(\theta_{x\to z}\right)\cos\left(\theta_{y\to z}\right)+\sin\left(\theta_{x\to y}\right)\sin\left(\theta_{y\to z}\right) & -\sin\left(\theta_{x\to y}\right)\sin\left(\theta_{x\to z}\right)\cos\left(\theta_{y\to z}\right)-\cos\left(\theta_{x\to y}\right)\sin\left(\theta_{y\to z}\right) & \cos\left(\theta_{x\to z}\right)\cos\left(\theta_{y\to z}\right)\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}r\\ 0\\ 0\end{pmatrix}$$

Given $r=\sqrt{x^2+y^2+z^2}$, we choose $\theta_{x\to y}=\arctan\left(\dfrac{y}{x}\right)$, $\theta_{x\to z}=\arctan\left(\dfrac{z}{\sqrt{x^2+y^2}}\right)$, and $\theta_{y\to z}=0$.

After applying $\theta_{y\to z}=0$, we have:

$$\boxed{\begin{pmatrix} \cos(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to z}) \\ -\cos(\theta_{x\to y})\sin(\theta_{x\to z})\sin(\theta_{y\to z}) - \sin(\theta_{x\to y})\cos(\theta_{y\to z}) & -\sin(\theta_{x\to y})\sin(\theta_{x\to z})\sin(\theta_{y\to z}) + \cos(\theta_{x\to y})\cos(\theta_{y\to z}) & \cos(\theta_{x\to z})\sin(\theta_{y\to z}) \\ -\cos(\theta_{x\to y})\sin(\theta_{x\to z})\cos(\theta_{y\to z}) + \sin(\theta_{x\to y})\sin(\theta_{y\to z}) & -\sin(\theta_{x\to y})\sin(\theta_{x\to z})\cos(\theta_{y\to z}) - \cos(\theta_{x\to y})\sin(\theta_{y\to z}) & \cos(\theta_{x\to z})\cos(\theta_{y\to z}) \end{pmatrix}}$$

$$\begin{pmatrix} \cos(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to z}) \\ -\cos(\theta_{x\to y})\sin(\theta_{x\to z})\sin(0) - \sin(\theta_{x\to y})\cos(0) & -\sin(\theta_{x\to y})\sin(\theta_{x\to z})\sin(0) + \cos(\theta_{x\to y})\cos(0) & \cos(\theta_{x\to z})\sin(0) \\ -\cos(\theta_{x\to y})\sin(\theta_{x\to z})\cos(0) + \sin(\theta_{x\to y})\sin(0) & -\sin(\theta_{x\to y})\sin(\theta_{x\to z})\cos(0) - \cos(\theta_{x\to y})\sin(0) & \cos(\theta_{x\to z})\cos(0) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to z}) \\ -\cos(\theta_{x\to y})\sin(\theta_{x\to z})(0) - \sin(\theta_{x\to y})(1) & -\sin(\theta_{x\to y})\sin(\theta_{x\to z})(0) + \cos(\theta_{x\to y})(1) & \cos(\theta_{x\to z})(0) \\ -\cos(\theta_{x\to y})\sin(\theta_{x\to z})(1) + \sin(\theta_{x\to y})(0) & -\sin(\theta_{x\to y})\sin(\theta_{x\to z})(1) - \cos(\theta_{x\to y})(0) & \cos(\theta_{x\to z})(1) \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \cos(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to z}) \\ -\sin(\theta_{x\to y}) & \cos(\theta_{x\to y}) & 0 \\ -\cos(\theta_{x\to y})\sin(\theta_{x\to z}) & -\sin(\theta_{x\to y})\sin(\theta_{x\to z}) & \cos(\theta_{x\to z}) \end{pmatrix}}$$

Then after applying $\theta_{x\to y} = \arctan\left(\frac{y}{x}\right)$ and $\theta_{x\to z} = \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)$, since $\cos\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{y}{\sqrt{x^2+y^2}}$, $\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, and $\sin\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) = \frac{z}{\sqrt{x^2+y^2+z^2}}$, we have:

$$\boxed{\begin{pmatrix} \cos(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to y})\cos(\theta_{x\to z}) & \sin(\theta_{x\to z}) \\ -\sin(\theta_{x\to y}) & \cos(\theta_{x\to y}) & 0 \\ -\cos(\theta_{x\to y})\sin(\theta_{x\to z}) & -\sin(\theta_{x\to y})\sin(\theta_{x\to z}) & \cos(\theta_{x\to z}) \end{pmatrix}}$$

$$\begin{pmatrix} \cos\left(\arctan\left(\frac{y}{x}\right)\right)\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) & \sin\left(\arctan\left(\frac{y}{x}\right)\right)\cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) & \sin\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) \\ -\sin\left(\arctan\left(\frac{y}{x}\right)\right) & \cos\left(\arctan\left(\frac{y}{x}\right)\right) & 0 \\ -\cos\left(\arctan\left(\frac{y}{x}\right)\right)\sin\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) & -\sin\left(\arctan\left(\frac{y}{x}\right)\right)\sin\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) & \cos\left(\arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right)\right) \end{pmatrix}$$

$$\begin{pmatrix} \left(\frac{x}{\cancel{\sqrt{x^2+y^2}}}\right)\left(\frac{\cancel{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2+z^2}}\right) & \left(\frac{y}{\cancel{\sqrt{x^2+y^2}}}\right)\left(\frac{\cancel{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2+z^2}}\right) & \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ -\left(\frac{y}{\sqrt{x^2+y^2}}\right) & \left(\frac{x}{\sqrt{x^2+y^2}}\right) & 0 \\ -\left(\frac{x}{\sqrt{x^2+y^2}}\right)\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) & -\left(\frac{y}{\sqrt{x^2+y^2}}\right)\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) & \left(\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right) \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ -\frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & -\frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \end{pmatrix}}$$

Therefore, since $r = \sqrt{x^2+y^2+z^2}$, we have:

$$\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ -\frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & -\frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x^2}{\sqrt{x^2+y^2+z^2}} + \frac{y^2}{\sqrt{x^2+y^2+z^2}} + \frac{z^2}{\sqrt{x^2+y^2+z^2}} \\ -\cancel{\frac{xy}{\sqrt{x^2+y^2}}} + \cancel{\frac{xy}{\sqrt{x^2+y^2}}} + 0 \\ -\frac{x^2z}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} - \frac{y^2z}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} + \frac{\left(\sqrt{x^2+y^2}\right)z}{\sqrt{x^2+y^2+z^2}} \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} \\ 0 \\ \frac{-x^2z - y^2z + \left(x^2+y^2\right)z}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{x^2+y^2+z^2} \\ 0 \\ \frac{-\cancel{x^2}z - \cancel{y^2}z + \cancel{x^2}z + \cancel{y^2}z}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{x^2+y^2+z^2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$